FREQUENCY DOMAIN ANALYSIS

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BOOKS

1. AUTOMATIC CONTROL SYSTEM *KUO & GOLNARAGHI*
2. CONTROL SYSTEM *ANAND KUMAR*
3. AUTOMATIC CONTROL SYSTEM S.HASAN SAEED

FREQUENCY DOMAIN ANALYSIS

FREQUENCY RESPONSE:

- The magnitude and phase relationship between sinusoidal input and steady state output of a system is known as frequency response.
- It is independent of the amplitude and phase of the input signal.
- If input signal is $r(t) = X\sin\omega t$ then output can be written as

$$c(t) = Y\sin(\omega t + \theta)$$

ADVANTAGES OF FREQUENCY RESPONSE:

- Frequency response tests are simple to perform.
- Transfer function can be obtained from frequency response of the system.
- Frequency response methods can be used to find the absolute as well as relative stability of the system.
- The apparatus required for obtaining frequency response are simple and easy to use.
- The effect of noise disturbance and parameter variations are relatively easy to visualize and assess through frequency response.

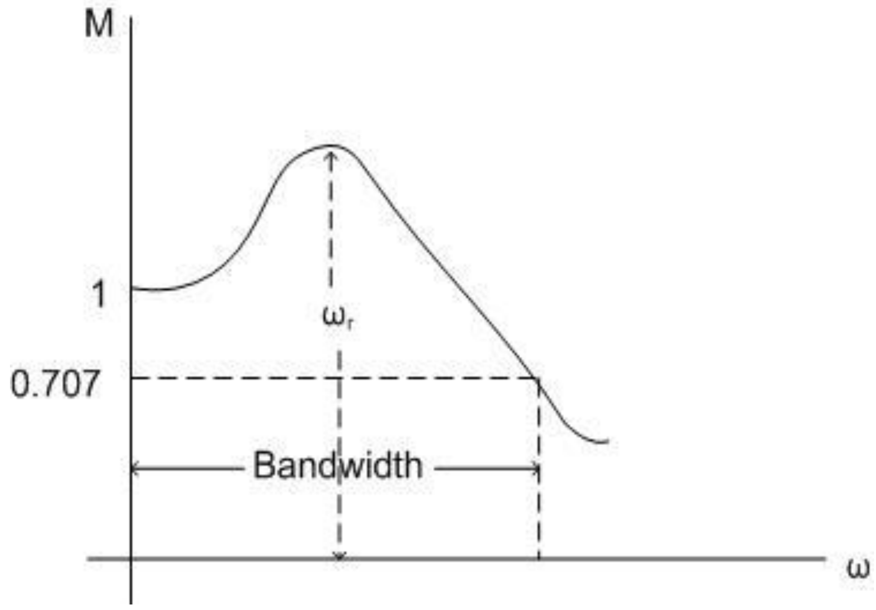
DISADVANTAGES OF FREQUENCY RESPONSE

- Obtaining frequency response practically is fairly time consuming.
- These methods are applied to linear systems.
- Frequency response test cannot be performed on non-interruptable systems.
- For high time constant, frequency response method is not convenient.

FREQUENCY DOMAIN SPECIFICATIONS

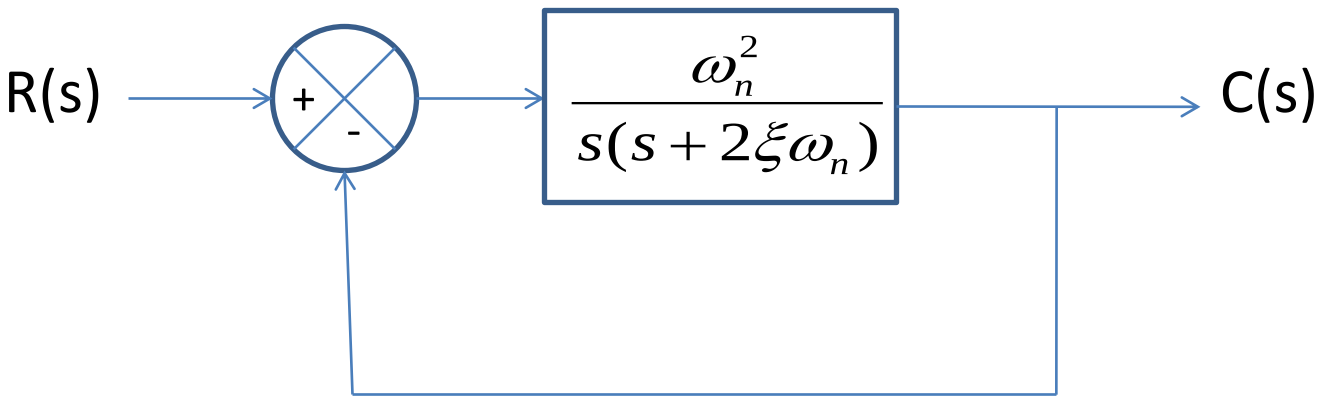
- **Resonant peak(M_r):** The maximum value of magnitude is known as resonant peak. The magnitude of resonant peak gives the information about the relative stability.
- **Resonant Frequency (ω_r):** The frequency at which magnitude has maximum frequency is known as resonant frequency.
- **Bandwidth:** Bandwidth is defined as the range of frequencies in which the magnitude of closed loop does not drop -3db. Or bandwidth is defined as the band of frequencies lying between -3db points.

- **Cut-off Frequency:** The frequency at which the magnitude is 3db below its zero frequency value is called cut-off frequency (ω_b).
- **Cut-off Rate:** The cut-off rate is the slope of the log magnitude curve near the cut off frequency.



Correlation between Time and Frequency Response

Consider the second order system



$$\frac{C(s)}{R(s)}=\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2} \quad ------(1)$$

Put s=jω in equation (1)

$$\frac{C(j\omega)}{R(j\omega)} = \frac{\omega_n^2}{(j\omega)^2 + 2\xi\omega_n(j\omega) + \omega_n^2} = \frac{1}{\left(1 - \frac{\omega^2}{\omega_n^2}\right) + j2\xi\frac{\omega}{\omega_n}}$$

Let $\frac{\omega}{\omega_n} = u$

$$\frac{C(j\omega)}{R(j\omega)} = \frac{1}{(1-u^2) + j2\xi u}$$

$$M(j\omega) = \frac{C(j\omega)}{R(j\omega)} = \frac{1}{\sqrt{(1-u^2)^2 + (2\xi u)^2}} ------(2)$$

$$\theta = -\tan^{-1}\frac{2\xi u}{1-u^2} ----(3)$$

The steady state output for a sinusoidal input of unit magnitude and variable frequency

$$C(t) = \frac{1}{\sqrt{(1-u^2)^2+(2\xi u)^2}}\sin\left[\omega t - \tan^{-1}\frac{2\xi u}{1-u^2}\right] ----(4)$$

From equation (2) & (3)

u	M	θ
0	1	0^0
1	$\frac{1}{2\xi}$	$-\frac{\pi}{2}$
∞	0	$-\pi$

The frequency at which 'M' has maximum value is known as resonant frequency ω_r

Put $u_r = \frac{\omega_r}{\omega_n}$

Where u_r is normalized resonant frequency

Differentiate equation (2) w.r.t u and put $u = u_r$

We get
$$4u_r^3 - 4u_r + 8u_r\xi^2 = 0$$
$$u_r = \sqrt{1-2\xi^2}$$
$$u_r = \frac{\omega_r}{\omega_n}$$
$$\omega_r = \omega_n\sqrt{1-2\xi^2} ------(5)$$

from equation (3)
$$\theta = -\tan^{-1}\frac{2\xi u_r}{1-u^2}$$
$$\therefore \theta = -\tan^{-1}\frac{\sqrt{1-2\xi^2}}{\xi}$$

For maximum value of magnitude put in equation (2)

$$u_r = \sqrt{1-2\xi^2}$$

$$\therefore M_r = \frac{1}{2\xi\sqrt{1-\xi^2}} ------(6)$$

For step response of second order system, maximum overshoot is given by

$$M_p = e^{-\frac{\pi\xi}{\sqrt{1-\xi^2}}} ------(7)$$

Damped frequency of oscillation is given by

$$\omega_d = \omega_n \sqrt{1-\xi^2} ------(8)$$

From equation (7) & (8)

$$\omega_r = \omega_d \left[\frac{\sqrt{1-2\xi^2}}{\sqrt{1-\xi^2}} \right] ------(9)$$

From above equations it is clear that if the value of ξ is small, the damped frequency ω_d and resonant frequency are nearly same. Hence for large value of ω_r the time response is faster.

As ξ increases, M_p decreases and gets vanished when $\xi = 1$. After that, system does not produce any overshoot. While in frequency domain M_r will Vanish if

$$\sqrt{1-2\xi^2} = 0$$

$$\therefore \xi = 0.707$$

and in such case system will not exhibit resonant peak.

When is very small i.e less than 0.4, both M_p and M_r will be very large and are not desirable. So ξ is generally designed to be between $0.4 < \xi < 0.707$

BANDWIDTH: The bandwidth is defined as the range of frequencies over which M is equal or greater than 0.707

From equation (2)

$$M = \frac{1}{\sqrt{(1-u_b^2)^2 + (2\xi u)^2}} = \frac{1}{\sqrt{2}}$$

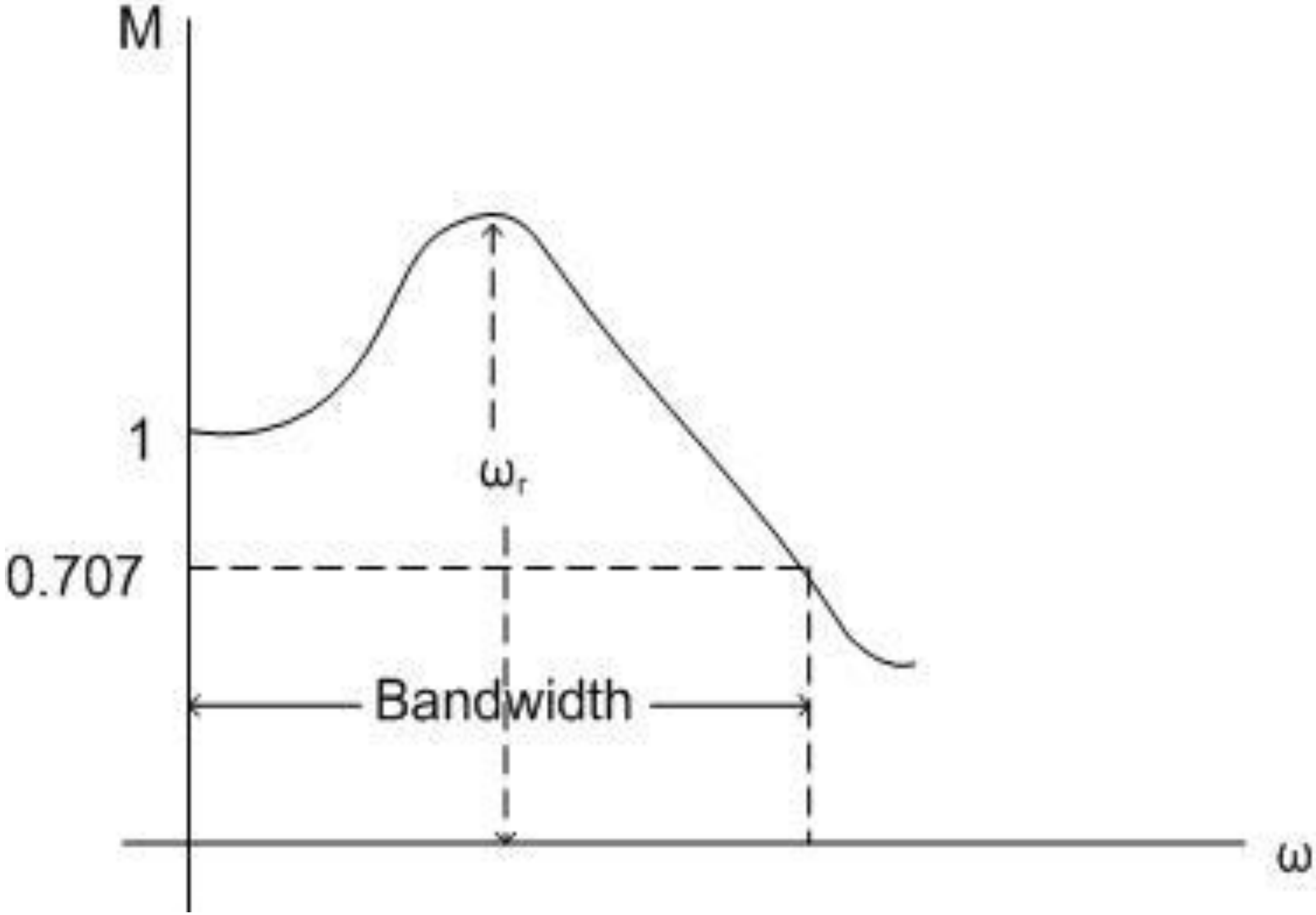
$$u_b^4 - 2u_b^2(1-2\xi^2) - 1 = 0$$

$$u_b = \sqrt{1-2\xi^2 + \sqrt{2-4\xi^2+4\xi^4}}$$

$$u_b = \frac{\omega_b}{\omega_n}$$

$$\omega_b = \omega_n\sqrt{1-2\xi^2 + \sqrt{2-4\xi^2+4\xi^4}}$$

M Vs ω



Example: The forward path transfer function of a unity feedback control system is

$$G(s) = \frac{100}{s(s+6.54)}$$

Find the resonance peak, resonant frequency and bandwidth of the closed loop system.

Solution: Given that

$$G(s) = \frac{100}{s(s+6.54)}$$

$$H(s) = 1$$

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

$$\frac{C(s)}{R(s)} = \frac{100}{s^2+6.54s+100}$$

Compare with $\frac{C(s)}{R(s)}=\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$

$$\omega_n^2=100$$

$$\omega_n=10rad/\sec$$

$$2\xi\omega_n=6.54$$

$$2\xi*10=6.54$$

$$\xi=0.327$$

$$\omega_r=\omega_n\sqrt{1-2\xi^2}$$

$$\omega_r=10\sqrt{1-2(0.327)^2}=8.86rad/\sec$$

$$M_r = \frac{1}{2\xi\sqrt{1-\xi^2}}$$

$$M_r = \frac{1}{2(0.327)\sqrt{1-(0.327)^2}}$$

$$M_r = 1.618$$

$$\omega_b = \omega_n\sqrt{1-2\xi^2+(2-4\xi^2+4\xi^4)^{1/2}}$$

$$\omega_b = 10\sqrt{1-2(0.327)^2+(2-4*0.327^2+4*0.327^4)^{1/2}}$$

$$\omega_b = 14.34 rad/\sec$$

CONSTANT MAGNITUDE CIRCLE (M-CIRCLE)

Let $G(j\omega) = x + jy$

Then from $\dfrac{C(j\omega)}{R(j\omega)} = \dfrac{G(j\omega)}{1+(j\omega)}$

$$M = M(j\omega) = \frac{|C(j\omega)|}{|R(j\omega)|} = \left|\frac{G(j\omega)}{1+G(j\omega)}\right| = \frac{|x+jy|}{|1+x+jy|} = \frac{\sqrt{x^2+y^2}}{\sqrt{(1+x)^2+y^2}} ----(1)$$

$$M^2 = \frac{x^2+y^2}{(1+x)^2+y^2}$$

$$M^2\left[(1+x)^2+y^2\right] = x^2+y^2$$

$$M^2\left[1+2x+x^2+y^2\right] = x^2+y^2$$

$$x^2(1-M^2)+y^2(1-M^2)-2xM^2=M^2$$

Divide both side by $(1-M^2)$

We get $x^2-y^2-2x\frac{M^2}{1-M^2}=\frac{M^2}{1-M^2}$

$\left[\frac{M^2}{1-M^2}\right]^2$ Add both sides

We get

$$\left[x-\frac{M^2}{1-M^2}\right]^2+(y-0)^2=\frac{M^2}{(1-M^2)^2}------(2)$$

Equation (1) is the equation of the circle with centre $\left[\frac{M^2}{1-M^2}, 0\right]$ and radius $\left[\frac{M}{1-M^2}\right]$

If M=1 then equation (1) becomes

$$(1+x)^2 + y^2 = x^2 + y^2$$

$$x = -\frac{1}{2}$$

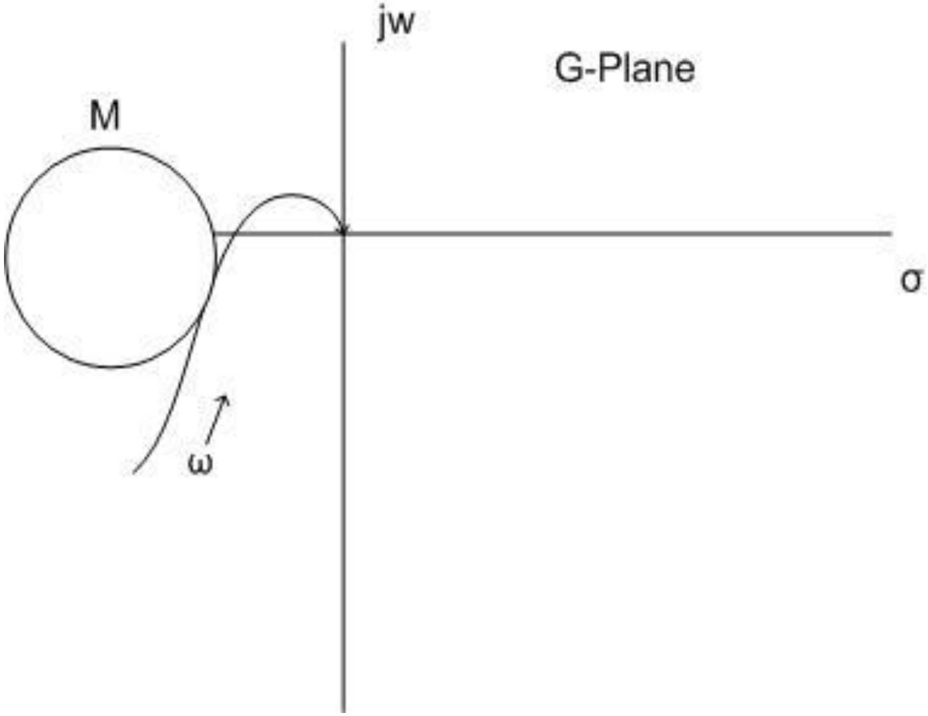
This is the equation of the straight line parallel to the y-axis and passing through (-1/2,0) in G(jω) plane.

The constant M locii for different values of M is shown in fig, it is clear that

(a) The locii are symmetrical with respect to M=1

(b) The M-circle for M>1 are on left side of the line M=1 and for M<1 the circles are on right side of the line M=1

(c) The intersection of G(jw) plot (Nyquist plot) and constant M locii gives the value of magnitudthe M-circle which is tangent to the G(jw) plot give the value of resonance peak and resonant frequency.


jw
G-Plane
M
σ
ω

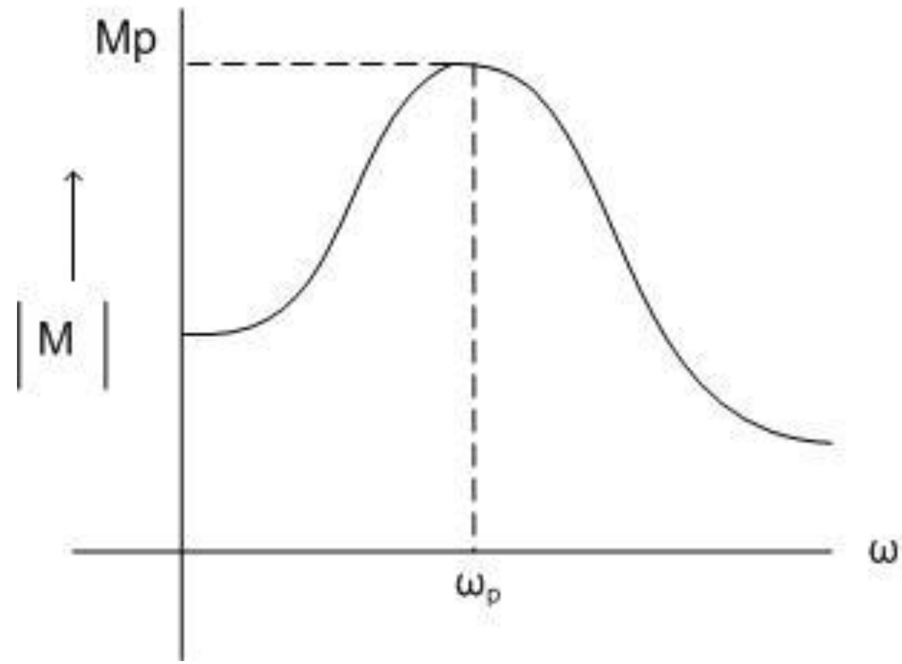
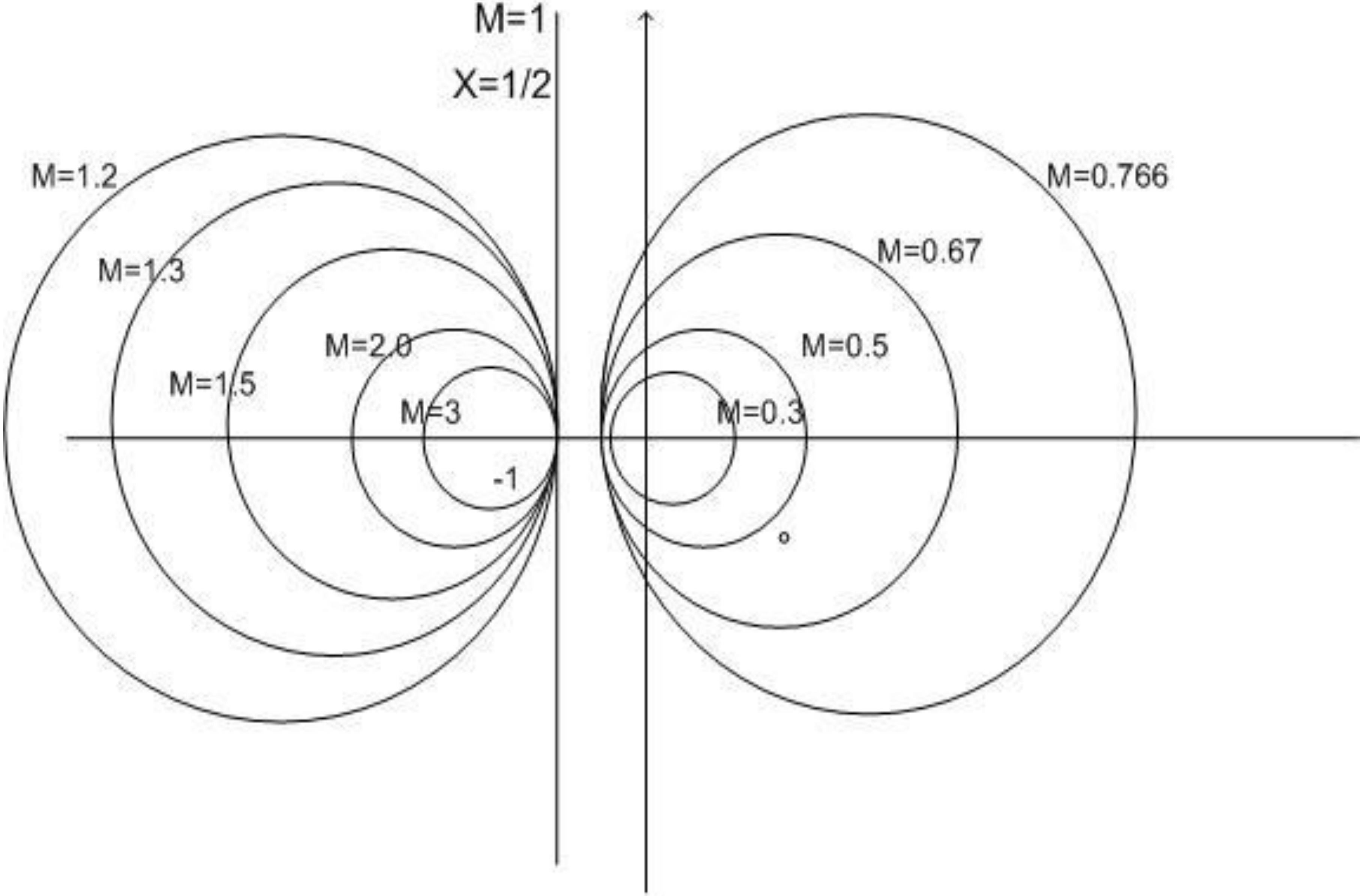

Mp
|M|
ω
ω_p

TABLE FOR CONSTRUCTION OF M-CIRCLE

S.NO.	M	CENTRE $\left(\frac{M^2}{1-M^2},0\right)$	RADIUS $\frac{M}{1-M^2}$
1.	0.3	(0.098,0)	0.329
2.	0.5	(0.33,0)	0.666
3.	0.67	(0.814,0)	1.215
4.	0.766	(1.42,0)	1.854
5.	0.833	(2.27,0)	2.72
6.	1.2	(-3.27,0)	-2.72



CONSTANT M-CIRCLES

CONSTANT N-CIRCLES (PHASE ANGLE LOCII

From equation (1), the phase shift can be written as

$$\phi = \tan^{-1}\frac{y}{x} - \tan^{-1}\frac{y}{1+x}$$

$$\tan\phi = \frac{\frac{y}{x} - \frac{y}{1+x}}{1+\left[\frac{y}{x}\right]\left[\frac{y}{1+x}\right]} = \frac{y}{x^2+x+y^2}$$

let

$$\tan\phi = N$$

$$N = \frac{y}{x^2+x+y^2}$$

$$x^2+x+y^2-\frac{y}{N}=0$$

Add to $\frac{1}{4}+\frac{1}{(2N)^2}$ both sides, we get

$$(x+1/2)^2+\left[y-\frac{1}{2N}\right]^2=\frac{1}{4}+\frac{1}{(2N)^2}$$

This equation represents the family of circles, with centers at (-1/2,1/2N) and radius $\sqrt{\frac{1}{4}+\frac{1}{4N^2}}$

Centre (-1/2,1/2N)

Radius= $\sqrt{\frac{N^2+1}{4N^2}}$

For different values of ϕ, N circles are shown in next slide

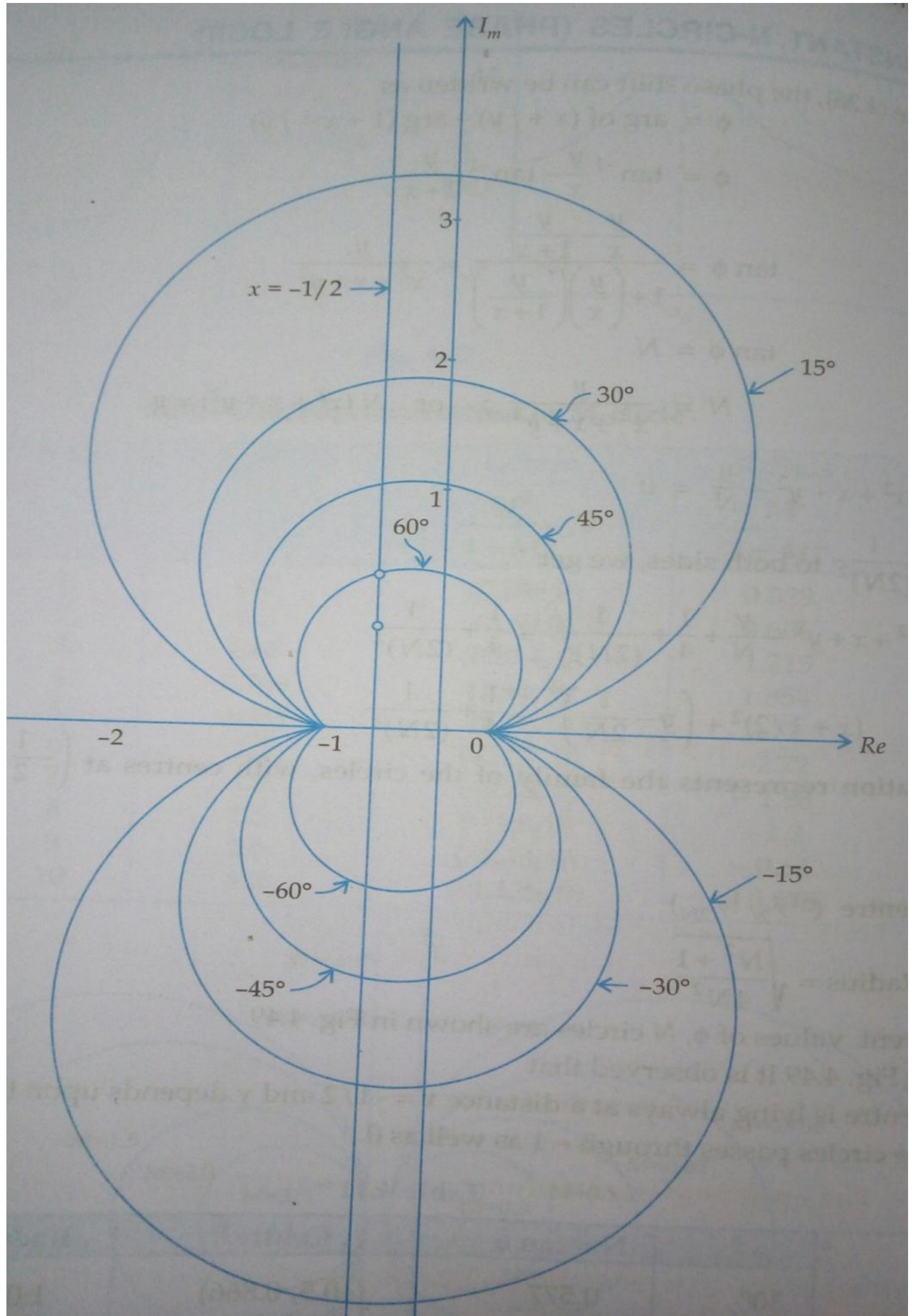
It is observed that

(1) The centre is lying always at a distance $x=-1/2$ and y depends upon the phase shift.

(2) All the circles passes through -1 as well as 0.

<u>Note: for different values of ϕ, calculate the value of N by $N=\tan\phi$, then find out centre and radius.</u>

CONSTANT N-CIRCLES



THANK YOU